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Separability and Vanishing Externalities

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In one of the most influential papers written on the subject of externalities, Otto Davis and Andrew Whinston argue that corrective taxes and private bargaining are likely to achieve an optimum in the presence of mutual externalities between two firms only when externalities are separable, in the sense that marginal cost is independent of the level of externality. Further analysis of the concept of separability reveals that even this conclusion is too optimistic. I shall argue below that the assumptions needed to make taxes and negotiations work properly rule out the possibility of having externalities in any observable situation.

The crucial step in Davis and Whinston's argument is the proposition that separability "... implies the game theoretic concept of dominance" (p. 247). They contend that when the cost function is separable, the profit-maximizing output of a firm affected by an externality is determined independently of the output choice of the firm causing the externality. Thus, when the process of mutual accommodation to externality is formulated as a noncooperative game, choosing that output level is a dominant strategy for the affected firm. Moreover, when the cost function is nonseparable, each firm must base its strategy on expectations about how the other firm will respond to its choices, and the game may fail to have an equilibrium point.

As stated by Davis and Whinston, the argument that separability of the cost function implies the existence of a dominant strategy is incorrect. A strategy should specify both inputs and outputs. In general, the functional relation between inputs and outputs depends on the level of externality. If the input combination which minimizes the cost to one firm of producing a fixed level of output changes as the output of the other firm changes, then it will not be true that the complete optimal strategy for one firm is independent of the actions of the other. The simplest example is the one input, one output case. Suppose an output Y^* is optimal for firm 1 under all actions of firm 2, but that the input required to obtain this output varies as the output of firm 2 varies. Then firm 1 cannot decide on an input-output pair without knowledge of what firm 2 will do.

Separability of the cost function implies only that the output choice is independent of externality. A further condition is needed to guarantee that input choice is also independent. For input choice to be independent of externality, the marginal productivity of each input must be independent of the level of externality. If this is to be the case, the production function must also be separable. Writers following Davis and Whinston, including James Marchand and Keith Russell, have assumed that separability of the cost function is equivalent to separability of the production function. It will be established that this assumption is false, and that it is only possible to have separability of both functions in a special case. I will give two counter examples.

Let $C(Y_1, Y_2)$ be the cost function of a firm which produces Y_1 and suffers an external diseconomy which is a function of Y_2. The cost function is dual to a production function $F(X_1, \ldots, X_n, Y_2)$.

Definition 1: A cost function $C(Y_1, Y_2)$ is separable if and only if it can be written as $C_1(Y_1)+C_2(Y_2)$.

Definition 2: A production function $F(X_1, \ldots, X_n, Y_2)$ is separable if and only if it can be written as $g(X_1, \ldots, X_n)+h(Y_2)$.

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LEMMA 1: *A cost function is separable if and only if* $\partial^2 C/\partial Y_1 \partial Y_2 = 0$ *everywhere. A production function is separable if and only if* $\partial^2 F/\partial X_i \partial Y_2 = 0, i = 1 \ldots n$, *everywhere.*[1]

Consider the separable production function

$$F = X_1^\alpha X_2^\beta - h(Y_2)$$

where $\alpha + \beta < 1$. We find the cost function by solving the cost-minimization problem and using the first-order conditions and the production function to eliminate the inputs from the cost equation. From the first-order conditons we have

$$\frac{W_1}{W_2} = \frac{\alpha X_2}{\beta X_1} \tag{1}$$

Solving for X_1 and substituting in the production function gives

$$Y_1 = X_2^{\alpha+\beta} \left(\frac{W_2 \alpha}{W_1 \beta}\right)^\alpha - h(Y_2) \tag{2}$$

Solving (2) for X_2, and substituting the resulting expression for X_2 in (1) enables us to express X_1 and X_2 in terms of Y_1 and Y_2 alone. Substitution in $C = W_1 X_1 + W_2 X_2$ gives the cost function

$$C = W_2 \left(1 + \frac{\alpha}{\beta}\right) \left(\frac{W_2 \alpha}{W_1 \beta}\right)^{-\alpha/(\alpha+\beta)} \cdot (Y_1 + h(Y_2))^{1/(\alpha+\beta)} \tag{3}$$

Clearly (3) is not separable if $\alpha + \beta \neq 1$.

Now consider the nonseparable production function

$$F = (X_1 - h_1(Y_2))^\alpha (X_2 - h_2(Y_2))^\beta$$

A mathematical development similar to that of the first example yields the separable cost function

$$C = W_2 \left(1 + \frac{\alpha}{\beta}\right) \left(\frac{\alpha}{\beta} \frac{W_2}{W_1}\right)^{-\alpha/(\alpha+\beta)} Y_1^{1/(\alpha+\beta)} + W_1 h_1(Y_2) + W_2 h_2(Y_2)$$

To find conditions under which the production function and its dual cost function are both separable, we express $\partial^2 C/\partial Y_1 \partial Y_2$ in terms of the derivatives of the production function. We adopt the following abbreviations

$$\frac{\partial F}{\partial X_i} = F_i \qquad \frac{\partial F}{\partial Y_2} = F_Y$$

$$\frac{\partial^2 F}{\partial X_i \partial X_j} = F_{ij} \qquad \frac{\partial^2 F}{\partial X_i \partial Y_2} = F_{iY}$$

Let

$$\Delta = \begin{vmatrix} F_{11} & \ldots & F_{1n} & F_1 \\ \vdots & & \vdots & \vdots \\ F_{n1} & \ldots & F_{nn} & F_n \\ F_1 & \ldots & F_n & 0 \end{vmatrix}$$

Further let Δ_{ij} be the i, jth cofactor of Δ.

THEOREM 1. *A cost function is separable if and only if it is derived from a production function which satisfies*

$$\sum_i F_{iY} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_Y = 0 \tag{4}$$

at every point which is a proper cost minimum.[2]

It follows that the proper statement of the relation between separability and dominance is that separability of the cost function *and* of the production function implies the game-theoretic concept of dominance. Moreover, the two separability hypotheses are satisfied simultaneously only when the private part g of the production function has a specific form.

COROLLARY 1: *If the cost function* $C(Y_1, Y_2)$ *and the production function* $F(X_1, \ldots, X_n, Y_2)$ *are both separable and* $F_Y \neq 0$ *everywhere, then* $|g_{ij}| = 0$, *where* $|g_{ij}|$ *is the Hessian determinant of* g.

PROOF:

By separability of the cost function (4) holds. Separability of the production function implies $F_{iY} = 0$ for all i. Therefore

[1] Lester Ford, p. 251, proves sufficiency. Necessity is trivial.

[2] For proof, see Appendix.

$$\Delta_{n+1,n+1} = \left| g_{ij} \right| = 0$$

COROLLARY 2: *If the cost function and production function are separable and* $F_Y \neq 0$, *then marginal cost* $\partial C/\partial Y_1$ *is constant in* Y_1 *and* Y_2.

PROOF:

From the Appendix and Paul Samuelson, p. 67,

$$\frac{\partial^2 C}{\partial Y_1^2} = \frac{\partial \lambda}{\partial Y_1} = \lambda \frac{\Delta_{n+1,n+1}}{\Delta} = 0$$

Constancy of marginal cost has some startling consequences. If no externality were present, then the supply function would be such that for any price of output greater than $\partial C/\partial Y_1$ the firm will produce unbounded output. At any price less than $\partial C/\partial Y_1$, it will produce zero output. Only when price of output equals $\partial C/\partial Y_1$ will the firm produce finite output. But when $C_2(Y_2) \neq 0$ there is a fixed cost, imposed on the firm by externality. Therefore even if price is exactly equal to $\partial C/\partial Y_1$ (which is a constant function of Y_1 and independent of Y_2), the firm will be losing money and will produce zero output But if price is at all higher than $\partial C/\partial Y_1$, the firm can earn unbounded profits by producing infinite output. Thus no equilibrium involving finite, nonzero output by a firm which suffers from an externality and has separable cost and production functions can exist unless the firm which causes the externality is producing zero output.

That is, when the conditions for the existence of dominant strategies in a two-firm externality game are satisfied, the only equilibrium possible is one in which one of the firms is out of business. It follows that it is impossible ever to observe a firm with separable cost and production functions suffering an externality, since either it or the firm causing the externality will always be driven out of business in equilibrium. If each firm creates an externality affecting the other, then only one can survive in equilibrium, and it will not suffer or cause any externality in equilibrium. Moreover, if the two firms produce the same output from the same input, this equilibrium is a Pareto optimum. Since the marginal costs of all firms are constant by hypothesis, one firm can produce any output as efficiently as many. The fixed cost imposed by externality implies that when more than one firm is in operation, more input is needed to produce a given output than is needed when only one firm operates.

The consequence of this analysis is the intensification of the pessimism expressed by Davis and Whinston regarding the possibility of using either corrective taxes or private bargaining to correct externalities. The only circumstances in which they thought such policies workable define a vacuous case, one in which externalities will never be observed or need to be remedied.

Appendix

Proof of Theorem 1

By Lemma 1 the cost function is separable if and only if $\partial^2 C/\partial Y_1 \partial Y_2 = 0$. We express $\partial^2 C/\partial Y_1 \partial Y_2$ in terms of the production function as follows. Form the Lagrangian expression

$$L = \sum W_i X_i + \lambda(Y_1 - F(X_1 \ldots X_n, Y_2))$$

First-order conditions are

$$W_i - \lambda F_i = 0$$

$$Y_1 - F = 0$$

We perturb the solution by varying Y_1 and Y_2. Totally differentiating the first-order conditions gives the system of equations

$$\text{(A1)} \quad \begin{pmatrix} F_{11} & \ldots & F_{1n} & F_1 \\ \vdots & & \vdots & \vdots \\ F_{n1} & \ldots & F_{nn} & F_n \\ F_1 & \ldots & F_n & 0 \end{pmatrix} \begin{pmatrix} dX_1 \\ \vdots \\ dX_n \\ d\lambda/\lambda \end{pmatrix} = \begin{pmatrix} \dfrac{dW_1}{\lambda} - F_{1Y} dY_2 \\ \vdots \\ \dfrac{dW_n}{\lambda} - F_{nY} dY_2 \\ dY_1 - F_Y dY_2 \end{pmatrix}$$

Solving for dX_k using Cramer's rule gives

$$dX_k \frac{1}{\Delta}\left\{ \sum_{i=1}^{n}\left[\left(\frac{dW_i}{\lambda} - F_{iY}dY_2\right)\Delta_{ik}\right] + (dY_1 - F_Y dY_2)\Delta_{n+1,k}\right\}$$

We assume that F is strictly quasi concave in $X_1 \ldots X_n$, so that $\Delta \neq 0$. Then

$$\frac{\partial X_k}{\partial Y_2} = \frac{-\sum_i F_{iY}\Delta_{ik} - F_Y\Delta_{n+1,k}}{\Delta}$$

Since

$$\frac{\partial C}{\partial Y_2} = \sum_k W_k \frac{\partial X_k}{\partial Y_2} \quad \text{and} \quad W_k = \lambda F_k,$$

$$\frac{\partial C}{\partial Y_2} = \frac{1}{\Delta}\left\{- \sum_{i=1}^{n}\left[F_{iY}\lambda\left(\sum_k F_k\Delta_{ik}\right)\right] - \lambda F_Y \sum_k F_k\Delta_{n+1,k}\right\}$$

But $\sum_k F_k\Delta_{n+1,k} = \Delta$, and $\sum_k F_k\Delta_{ik} = 0$ since it is an expansion by alien cofactors. Therefore

$$\text{(A2)} \qquad \frac{\partial C}{\partial Y_2} = -\lambda F_Y$$

Differentiating (A2) with respect to Y_1 gives

$$\frac{\partial^2 C}{\partial Y_2 \partial Y_1} = -\lambda \sum_i F_{iY}\frac{\partial X_i}{\partial Y_1} - F_Y\frac{\partial \lambda}{\partial Y_1}$$

From (A1),

$$\frac{\partial \lambda}{\partial Y_1} = \lambda \frac{\Delta_{n+1,n+1}}{\Delta}, \qquad \frac{\partial X_i}{\partial Y_1} = \frac{\Delta_{n+1,i}}{\Delta}$$

Therefore

$$\frac{\partial^2 C}{\partial Y_2 \partial Y_1} = -\frac{\lambda}{\Delta}\left(\sum_i F_{iY}\Delta_{n+1,i} + \Delta_{n+1,n+1}F_Y\right)$$

REFERENCES

O. Davis and A. Whinston, "Externalities, Welfare, and the Theory of Games," *J. Polit. Econ.*, June 1962, *70*, 241–62.

L. Ford, *Differential Equations*, New York 1955.

J. R. Marchand and K. P. Russell, "Externalities, Liability, Separability, and Resource Allocation," *Amer. Econ. Rev.*, Sept. 1973, *63*, 611–20.

P. A. Samuelson, *Foundations of Economic Analysis*, Cambridge, Mass. 1966.